J. Appl. Math. Comput. (2016) 50:299–314
DOI 10.1007/s12190-015-0871-0

ORIGINAL RESEARCH

A new parallel subspace correction method for advection–diffusion equation

Jiansong Zhang · Hongfei Fu · Hui Guo

Received: 2 December 2014 / Published online: 4 February 2015

Abstract A new parallel subspace correction algorithm is proposed to solve advection–diffusion equation with mass-conservative characteristic finite element (MCC-FE) procedure. The dependence relations of the subdomains overlapping size, spacial mesh parameter, time step, iteration number with the convergence rate is analyzed, and the a priori error estimate of this parallel algorithm is given. Some numerical experiments are given to verify our theoretical result.

Keywords MCC-FE · Domain decomposition · Subspace correction · Advection–diffusion equation

Mathematics Subject Classification 65M15 · 65M25 · 65M55 · 65M60

1 Introduction

Recently, parallel computation has been a powerful tool for solving a large scale partial differential equation systems. Overlapping domain decomposition method is

J. Zhang (✉)
Department of Applied Mathematics, China University of Petroleum, Qingdao 266580, China
e-mail: jszhang@upc.edu.cn

H. Fu
Department of Pure Mathematics, China University of Petroleum, Qingdao 266580, China
e-mail: hongfeifu@upc.edu.cn

H. Guo
Department of Computational Mathematics, China University of Petroleum, Qingdao 266580, China
e-mail: sdugh@163.com

a class of important numerical methods widely used in many fields. On the basis of overlapping domain decomposition, lots of numerical methods for elliptic problems have been developed. The earliest overlapping domain decomposition method is Schwarz alternating method. The traditional Schwarz alternating algorithms are not parallel but successive. In order to obtain parallel Schwarz alternating algorithms, many new techniques have been introduced, such as additive Schwarz methods, parallel multilevel precondition algorithms, parallel weighted Schwarz algorithms, parallel subspace correction methods and etc., see [1–8]. Overlapping domain decomposition methods are extended to the parabolic problems. Generally speaking, through using finite difference techniques in time, the parabolic problems can be changed into a set of elliptic problems at each time step. One can use any parallel overlapping domain decomposition algorithms, which are effective for elliptic problems, to solve these resulting elliptic problems step by step over time. Cai in [9,10] presented some additive Schwarz methods for parabolic problems but not gave the convergence analysis. Tai in [11] proposed parallel weighted Schwarz algorithms for solving parabolic equations and analyzed iterative number needed at each time step to reach given accuracy. Rui and Yang in [12,13] constructed and analyzed the traditional Schwarz algorithms of parabolic problems and gave a convergent rate that depends on mesh size. H. Wang, J. Liu etc. in [14] developed a characteristic quasi-two-level, coarse-mesh-free domain decomposition method for unsteady convection-diffusion equations. Sun, Yang [15] and Yang [16] proposed improved domain decomposition parallel methods for parabolic equations and derived an almost optimal error estimates, without the factor $H^{-\frac{1}{2}}$ given in Dawson-Dupont's error estimate in [17].

In this article, basing on subspace correction idea proposed by J. Xu [6], we propose a new overlapping domain decomposition parallel algorithm combined with the MCC-FE procedure to solve advection–diffusion equation. In this algorithm, by the similar techniques as in [18–20], we utilize the partition functions of unity to distribute the corrections in the overlapping domains reasonably. We study the dependence relations of the subdomains overlapping size, spatial mesh parameter, time step and iteration number with the convergence rate, and give the a priori error estimate. Theoretical analysis suggests that we only need one or two iterations to reach given accuracy at each time level.

The outline of this paper is as follows. Firstly, we revisit the MCC-FE procedure for the advection–diffusion equation and propose a new parallel algorithm in Sect. 2. Then, we give some important lemmas in Sect. 3, which will be used to complete the proof of the convergence theorem. In Sect. 4, we give the convergence analysis and prove convergence theorem. In Sect. 5, we present some numerical examples to verify theoretical results.

2 Formulation of parallel algorithm

To illustrate our method, we just consider the following advection–diffusion problem as our model: Let $\Omega = [a, b],\ t > 0$,

$$
\begin{aligned}
&\text{(a)}\quad \frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\nu u-\varepsilon\frac{\partial u}{\partial x}\right)=f(x,t),\\
&\text{(b)}\quad \left(\nu u-\varepsilon\frac{\partial u}{\partial x}\right)|_{x=a}=\left(\nu u-\varepsilon\frac{\partial u}{\partial x}\right)|_{x=b}=0,\\
&\text{(c)}\quad u(x,0)=u_0,
\end{aligned}
\tag{1}
$$

where $\nu=\nu(x,t)$ is a given velocity, $\varepsilon>0$ is a diffusion coefficient, f and $u^0(x)$ are given functions.

Hypothesis I [21] The velocity ν satisfies

$$\nu\in C^0([0,T];\ W^{1,\infty}(a,b)),\quad \nu(a,t)=\nu(b,t)=0.$$

For given $(x,t)\in[a,b]\times[0,T]$, define the characteristic line through (x,t) be the function $X(x,t;\tau)$ satisfying the following initial value problem

$$
\begin{cases}
\dfrac{dX}{d\tau}=\nu(X(x,t;\tau),\tau),\\
X(x,t;t)=x.
\end{cases}
\tag{2}
$$

Set $\psi=[1+\nu^2]^{\frac{1}{2}}$, then the equation (1a) can be rewritten in an equivalent form

$$\psi\frac{\partial u}{\partial\tau}+\frac{\partial\nu}{\partial x}u-\frac{\partial}{\partial x}\left(\varepsilon\frac{\partial u}{\partial x}\right)=f. \tag{3}$$

Let time increment $\triangle t>0$ and $t^n=n\triangle t$. According to (2), giving an initial condition $X(x,t^n;t^n)=x$, we can get an approximate value of X by the Euler method

$$X_1^n=X_1(x,t^n;t^{n-1})=x-\nu^n(x)\triangle t. \tag{4}$$

Proposition I [21] *Under Hypothesis I and*

$$\triangle t\le\frac{1}{\|\nu\|_{C^0(W^{1,\infty})}},$$

it holds

$$X_1^n([a,b])=[a,b].$$

To construct a new parallel algorithm, we firstly give a domain decomposition. Denote $\{\Omega_i'\}_{i=1}^N$ a non-overlapping domain decomposition of Ω. In order to obtain an overlapping domain decomposition, we extend each subregion Ω_i' to a larger region Ω_i such that $\Omega_i'\subset\Omega_i\subset\Omega$ and $dist(\partial\Omega_i'\backslash\partial\Omega,\partial\Omega_i\backslash\partial\Omega)\ge H$ for each $1\le i\le N$, where $H>0$ is called as overlapping degree. Let $\mathcal{T}_h$ be a family of quasi-regular finite element partition of the domain Ω such that the elements on the partition have the diameters bounded by h. Assume that $\mathcal{T}_{h,i}=\mathcal{T}_h\bigcap\Omega_i$ just is one finite element partition of Ω_i for $1 \le i \le N$. Let $\mathcal{V}_h \subset H_0^1(\Omega)$ be a piecewise k-degree polynomial space defined on the partition $\mathcal{T}_h$.

Define the following bilinear form:

$$A(u, v) = (u, v) + \Delta t(\varepsilon u_x, v_x).$$

Based on (3) and (4), the MCC-FE scheme can be read from [21,22] as follows: for $\forall v_h \in \mathcal{V}_h,\ n \ge 1$,

$$\begin{aligned} &\text{(a)} \quad A\big(\rho_h^n, v_h\big) = \big(\rho_h^{n-1} \circ X_1^n \gamma^n, v_h\big) + \Delta t\big(f^n, v_h\big), \\ &\text{(b)} \quad (\rho_h^0, v_h) = (u_0, v_h), \end{aligned} \tag{5}$$

where the definitions of $\rho_h^{n-1} \circ X_1^n$ and γ^n are as follows:

$$\begin{aligned} \big(\rho_h^{n-1} \circ X_1^n\big)(x) &= \rho_h^{n-1}\big(X_1^n(x)\big), \\ \gamma^n &= 1 - \Delta t \frac{\partial v^n}{\partial x}. \end{aligned}$$

In the following part of this section, we propose a new domain decomposition parallel algorithm of the system (1). Define finite element sub-spaces as follows:

$$\mathcal{V}_h^i = \big\{v_h \in \mathcal{V}_h;\ \ v_h = 0 \ \text{ in } \ \Omega \backslash \Omega_i\big\}, \quad 1 \le i \le N.$$

It is clear that

$$\mathcal{V}_h = \mathcal{V}_h^1 + \mathcal{V}_h^2 + \cdots + \mathcal{V}_h^N.$$

Obviously, there exists a finite open covering family $\{O^i\}_{i=1}^N$ of the domain Ω such that $O^i \cap \Omega \subset \Omega_i$. Using the theory of partition of unity, we know that there exists a function sequence $\{\varphi_i\}_{i=1}^N$ such that

$$\begin{aligned} &\text{(a)} \ \ supp(\varphi_i) \subset O^i, \quad 0 \le \varphi_i \le 1, \quad \|\varphi_i\|_{W^{r,\infty}} \le K H^{-r}, \ \ 1 \le i \le N, \\ &\text{(b)} \ \ \varphi_1 + \varphi_2 + \cdots + \varphi_N = 1 \ \text{ in } \ \Omega, \end{aligned}$$

where r is some nonnegative integer. Let φ_h^i be the piecewise linear interpolation of φ_i on $\mathcal{T}_h$, and $\mathcal{I}_h$ be the interpolation operator on $\mathcal{V}_h$. So, we propose a parallel subspace correction algorithm.

Parallel Algorithm Denote m the iteration number at each time level. For given initial approximation $u_h^0 = \rho_h^0 \in \mathcal{V}_h$, seek $u_h^n \in \mathcal{V}_h (n = 1,\ 2,\ \cdots)$, by four steps:

Step 1. Set $\tilde{u}_0^n = u_h^{n-1}$ and $j = 1$.
Step 2. For $i = 1, 2, \ldots, N$, seek $e_j^i \in \mathcal{V}_h^i$, in parallel, such that

$$\begin{aligned} A\big(e_j^i, v_h\big) = &\big(u_h^{n-1} \circ X_1^n \gamma^n, \mathcal{I}_h(\varphi_h^i v_h)\big) + \Delta t\big(f^n, \mathcal{I}_h(\varphi_h^i v_h)\big) \\ &- A\big(\tilde{u}_{j-1}^n, \mathcal{I}_h(\varphi_h^i v_h)\big), \quad \forall v_h \in \mathcal{V}_h^i. \end{aligned} \tag{6}$$

Step 3. Set

$$\tilde{u}_j^n = \tilde{u}_{j-1}^n + \sum_{i=1}^{N} e_j^i. \tag{7}$$

Step 4. If $j < m$, then set $j := j + 1$ and back to the step 2; or set $u_h^n = \tilde{u}_m^n$ and then return the first step to start iteration at the next time level.

For Parallel Algorithm, we have the following main result:

Theorem 1 *Let u and u_h^n be the solutions of* (1) *and Parallel algorithm, respectively. Then the a priori error estimate*

$$\max_n \|u^n - u_h^n\|_{L^2(\Omega)} \leq K \left\{ \left(\frac{h^2}{H^2} + \frac{\triangle t}{H^2} \right)^{\frac{m}{2}} + h^{k+1} + \triangle t \right\}, \tag{8}$$

holds, where K is a positive constants independent of the mesh parameters H, h and $\triangle t$.

3 Some important lemmas

Define the norm $\|\cdot\|_A$ as follows:

$$\|w\|_A^2 = \big(w, w\big) + \triangle t\big(\varepsilon w_x, w_x\big).$$

We introduce a projection operator $P_h^i : \mathcal{V}_h \to \mathcal{V}_h^i$ such that

$$A\big(P_h^i v, v_h\big) = A\big(v, v_h\big), \quad \forall\ v_h \in \mathcal{V}_h^i, \quad i = 1, 2 \ldots, N.$$

In order to analyze the convergence of Parallel algorithm, now we give some important lemmas.

Lemma 1 [21] *Let u and ρ_h be the solutions of* (1) *and* (5). *Then, the a priori estimate*

$$\|u - \rho_h\|_{L^\infty(L^2(\Omega))} + \varepsilon \|u - \rho_h\|_{L^2(H^1(\Omega))} \leq K\Big\{h^{k+1} + \triangle t\Big\} \tag{9}$$

holds.

Lemma 2 *For $s = 0, 1$, there holds*

$$\|(\mathcal{I} - \mathcal{I}_h)(\varphi_h^i v_h)\|_{H^s(\Omega)} \leq K \frac{h}{H} \|v_h\|_{H^s(\Omega)}, \quad \forall\, v_h \in \mathcal{V}_h, \quad 1 \leq i \leq N, \tag{10}$$

where $\mathcal{I}$ is an identity operator.

Proof It is easily seen that

$$\begin{aligned}\|(\mathcal{I}-\mathcal{I}_h)(\varphi_h^i v_h)\|_{H^s(\Omega)}^2 &\le Kh^{2(k+1-s)}\sum_{\kappa\in\mathcal{T}_h}|\varphi_h^i v_h|_{H^{k+1}(\kappa)}^2\\ &\le Kh^{2(k+1-s)}\sum_{\kappa\in\mathcal{T}_h}|D\varphi_h^i D^k v_h|_{L^2(\kappa)}^2\\ &\le Kh^2\|\varphi_h^i\|_{W^{1,\infty}(\Omega)}^2\|v_h\|_{H^s(\Omega)}^2\\ &\le K\frac{h^2}{H^2}\|v_h\|_{H^s(\Omega)}^2. \end{aligned} \tag{11}$$

where we have used the approximate properties of finite element space $\mathcal{V}_h$ and the fact $D^{k+1}v_h=0$ on each element. So we get the inequality of (10). ◻

Lemma 3 *The following estimate*

$$\left|A\big(w,v\big)-\sum_{i=1}^N A\big(w,\mathcal{I}_h(\varphi_h^i P_h^i v)\big)\right|\le K\left(\frac{h}{H}+\frac{\sqrt{\Delta t}}{H}\right)\|w\|_A\|v\|_A \tag{12}$$

holds for all $w,\ v\in\mathcal{V}_h$.

Proof It is easily seen that

$$A\big(w,\mathcal{I}_h(\varphi_h^i P_h^i v)\big)=A\big(w,\varphi_h^i P_h^i v\big)-A\big(w,(\mathcal{I}-\mathcal{I}_h)(\varphi_h^i P_h^i v)\big),$$

and

$$A\left(w,\varphi_h^i P_h^i v\right)=A\left(\varphi_h^i w,P_h^i v\right)+\Delta t\left[\left(\varepsilon w_x,\frac{\partial\varphi_h^i}{\partial x}P_h^i v\right)-\left(\varepsilon\frac{\partial\varphi_h^i}{\partial x}w,\frac{\partial P_h^i v}{\partial x}\right)\right].$$

Note that

$$A(w,v)=\sum_{i=1}^N A(\varphi_h^i w,v).$$

Hence, we have

$$\begin{aligned}&A\big(w,v\big)-\sum_{i=1}^N A\big(w,\mathcal{I}_h(\varphi_h^i P_h^i v)\big)\\ &=\sum_{i=1}^N A\big(\varphi_h^i w,(\mathcal{I}-P_h^i)v\big)+\sum_{i=1}^N A\big(w,(\mathcal{I}-\mathcal{I}_h)(\varphi_h^i P_h^i v)\big)\\ &-\Delta t\sum_{i=1}^N\left[\left(\varepsilon w_x,\frac{\partial\varphi_h^i}{\partial x}P_h^i v\right)-\left(\varepsilon\frac{\partial\varphi_h^i}{\partial x}w,\frac{\partial P_h^i v}{\partial x}\right)\right].\end{aligned} \tag{13}$$

Using Lemma 2, we obtain

$$\begin{aligned}\left|A\big(w, (\mathcal{I}-\mathcal{I}_h)(\varphi_h^i P_h^i v)\big)\right| &= \Bigg|(w, (\mathcal{I}-\mathcal{I}_h)(\varphi_h^i P_h^i v)) \\ &\quad + \triangle\, t\left(\varepsilon w_x, \frac{\partial(\mathcal{I}-\mathcal{I}_h)(\varphi_h^i P_h^i v)}{\partial x}\right)\Bigg| \\ &\le K\left(\frac{h}{H}+\frac{\sqrt{\Delta t}}{H}\right)\|w\|_{A,\Omega^i}\|v\|_{A,\Omega^i}.\end{aligned}$$

So, we have

$$\left|\sum_{i=1}^{N} A\left(w, (\mathcal{I}-\mathcal{I}_h)(\varphi_h^i P_h^i v)\right)\right| \le K\Big(\frac{h}{H}+\frac{\sqrt{\Delta t}}{H}\Big)\|w\|_A\|v\|_A,$$

and

$$\begin{aligned}\left|\sum_{i=1}^{N} A\left(\varphi_h^i w, (\mathcal{I}-P_h^i)v\right)\right| &= \left|\sum_{i=1}^{N} A\left((\mathcal{I}-\mathcal{I}_h)\varphi_h^i w, (\mathcal{I}-P_h^i)v\right)\right| \\ &\le K\left(\frac{h}{H}+\frac{\sqrt{\Delta t}}{H}\right)\left(\sum_{i=1}^{N}\|(\mathcal{I}-P_h^i)v\|_{A,\Omega^i}^2\right)^{1/2}\|w\|_A.\end{aligned}$$

In addition, we have

$$\begin{aligned}&\Delta t\sum_{i=1}^{N}\left|\left(\varepsilon w_x, \frac{\partial\varphi_h^i}{\partial x}P_h^i v\right)-\left(\varepsilon\frac{\partial\varphi_h^i}{\partial x}w, \frac{\partial P_h^i v}{\partial x}\right)\right| \\ &\le K\frac{\sqrt{\Delta t}}{H}\left(\sum_{i=1}^{N}\|P_h^i v\|_{A,\Omega^i}^2\right)^{1/2}\|w\|_A\end{aligned}$$

Substituting these inequalities into (13), we get the estimate (12). □

4 Convergence analysis

In this section, we give the complete proof of the convergence theorem.

We know that Parallel algorithm is equivalent to an iteration with initial value u_h^{n-1} to solve the following problem: Find $\hat{u}_h^n \in \mathcal{V}_h$ such that for any $v_h \in \mathcal{V}_h$

$$A(\hat{u}_h^n, v_h) = (u_h^{n-1}\circ X_1^n\gamma^n, v_h) + \Delta t(f^n, v_h). \tag{14}$$

We can rewrite (14) as follows:

$$A(u_h^n, v_h) = (u_h^{n-1} \circ X_1^n \gamma^n, v_h) + \triangle t(f^n, v_h) + A(u_h^n - \hat{u}_h^n, v_h). \tag{15}$$

Subtracting (5) from (15), we get the equation

$$A(u_h^n - \rho_h^n, v_h) = \big((u_h^{n-1} - \rho_h^{n-1}) \circ X_1^n \gamma^n, v_h\big) + A(u_h^n - \hat{u}_h^n, v_h). \tag{16}$$

It is obvious that we must estimate the bound of $u_h^n - \rho_h^n$ to bound $u_h^n - u^n$. The last equation suggests that the bound of $u_h^n - \hat{u}_h^n$ is crucial, which reflects the contribution of the iteration error to the global approximation error.

Lemma 4 *There exists the a priori estimate*

$$\|u_h^n - \hat{u}_h^n\|_A \leq K\left(\frac{h^2}{H^2} + \frac{\Delta t}{H^2}\right)^{\frac{m}{2}} \|u_h^{n-1} - \hat{u}_h^n\|_A. \tag{17}$$

Proof From Parallel algorithm and (14), we know that

$$A(e_j^i, v_h) = A(e_j^i, P_h^i v_h) = A(\hat{u}_h^n - \tilde{u}_{j-1}^n, \mathcal{I}_h(\varphi_h^i P_h^i v_h)), \tag{18}$$

and

$$\begin{aligned} A(\tilde{u}_j^n - \hat{u}_h^n, v_h) &= A(\tilde{u}_{j-1}^n - \hat{u}_h^n, v_h) + \sum_{i=1}^N A(e_j^i, v_h) \\ &= A(\tilde{u}_{j-1}^n - \hat{u}_h^n, v_h) + \sum_{i=1}^N A(\hat{u}_h^n - \tilde{u}_{j-1}^n, \mathcal{I}_h(\varphi_h^i P_h^i v_h)). \end{aligned} \tag{19}$$

By Lemma 3, we have the estimate

$$\|\tilde{u}_j^n - \hat{u}_h^n\|_A \leq K\left(\frac{h}{H} + \frac{\sqrt{\Delta t}}{H}\right) \|\tilde{u}_{j-1}^n - \hat{u}_h^n\|_A. \tag{20}$$

Hence, we have the inequality (17). □

Set $\xi^n = u_h^n - \rho_h^n$, $\eta^n = u_h^n - \hat{u}_h^n$ and $\zeta^n = \rho_h^n - u^n$. Now, we bounds $u_h^{n-1} - \hat{u}_h^n$. The following result can be obtained.

Lemma 5 *The a priori estimate*

$$\|\hat{u}_h^n - u_h^{n-1}\|_A^2 \leq K\Delta t\big\{1 + \|\xi^{n-1}\|_{H^1(\Omega)}^2\big\} \tag{21}$$

holds.

Proof Using (14) and the definition of γ^n, we get the equation

$$
\begin{aligned}
A(\hat{u}_h^n - u_h^{n-1}, v_h) &= (u_h^{n-1} \circ X_1^n \gamma^n, v_h) - A(u_h^{n-1}, v_h) + \Delta t(f^n, v_h) \\
&= \left(u_h^{n-1} \circ X_1^n \gamma^n - u_h^{n-1}, v_h\right) - \Delta t \left(\varepsilon \frac{\partial u_h^{n-1}}{\partial x}, \frac{\partial v_h}{\partial x}\right) \\
&\quad + \Delta t\left(f^n, v_h\right) \\
&= \left(u_h^{n-1} \circ X_1^n - u_h^{n-1}, v_h\right) - \Delta t \left(u_h^{n-1} \circ X_1^n \frac{\partial \nu^n}{\partial x}, v_h\right) \\
&\quad - \Delta t \left(\varepsilon \frac{\partial u_h^{n-1}}{\partial x}, \frac{\partial v_h}{\partial x}\right) + \Delta t \left(f^n, v_h\right) \\
&= \left(u_h^{n-1} \circ X_1^n - u_h^{n-1}, v_h\right) - \Delta t \left(u^{n-1} \circ X_1^n \frac{\partial \nu^n}{\partial x}, v_h\right) \\
&\quad - \Delta t \left(\zeta^{n-1} \circ X_1^n \frac{\partial \nu^n}{\partial x}, v_h\right) - \Delta t \left(\xi^{n-1} \circ X_1^n \frac{\partial \nu^n}{\partial x}, v_h\right) \\
&\quad - \Delta t \left(\varepsilon \frac{\partial u^{n-1}}{\partial x}, \frac{\partial v_h}{\partial x}\right) - \Delta t \left(\varepsilon \frac{\partial \zeta^{n-1}}{\partial x}, \frac{\partial v_h}{\partial x}\right) \\
&\quad - \Delta t \left(\varepsilon \frac{\partial \xi^{n-1}}{\partial x}, \frac{\partial v_h}{\partial x}) + \Delta t\left(f^n, v_h\right)\right).
\end{aligned}
\tag{22}
$$

Choosing $v_h = \hat{u}_h^n - u_h^{n-1}$ in (22) and using Lemma 1, we obtain

$$
\begin{aligned}
\|\hat{u}_h^n - u_h^{n-1}\|_A^2 \le{}& \|u_h^{n-1} \circ X_1^n - u_h^{n-1}\|_{H^{-1}(\Omega)} \|\hat{u}_h^n - u_h^{n-1}\|_{H^1(\Omega)} \\
&+ K(\Delta t)^2\left\{1 + \|\xi^{n-1}\|_{L^2(\Omega)}^2\right\} + K\Delta t\varepsilon\left\{1 + \|\xi^{n-1}\|_{H^1(\Omega)}^2\right\} \\
&+ \delta\|\hat{u}_h^n - u_h^{n-1}\|_A^2
\end{aligned}
\tag{23}
$$

where we have used Lemma 1 and the inequality (Lemma 1 in [23]) $\|\phi \circ X_1^n\| \le (1 + K\Delta t)\|\phi\|$.

Using the similar idea to Lemma 1 in [24] and Lemma 1, we have

$$
\begin{aligned}
\|u_h^{n-1} \circ X_1^n - u_h^{n-1}\|_{H^{-1}(\Omega)} &\le K\|u_h^{n-1}\|_{L^2(\Omega)} \Delta t \\
&\le K\{\|\xi^{n-1}\|_{L^2(\Omega)} + \|\zeta^{n-1}\|_{L^2(\Omega)} + \|u^{n-1}\|_{L^2(\Omega)}\}\Delta t \\
&\le K\{1 + \|\xi^{n-1}\|_{L^2(\Omega)}\}\Delta t.
\end{aligned}
$$

Substitute the above estimate into (23), then we get the inequality (21). □

Next, we complete the proof of Theorem 1.

Proof Choosing $v_h = \xi^n$ in (16), we have

$$
\left(\xi^n - \xi^{n-1} \circ X_1^n \gamma^n, \xi^n\right) + \Delta t\left(\varepsilon \xi_x^n, \xi_x^n\right) = A\left(\eta^n, \xi^n\right). \tag{24}
$$

We note that

$$\begin{aligned}
&\left(\xi^n - \xi^{n-1} \circ X_1^n \gamma^n, \xi^n\right) + \Delta t\left(\varepsilon \xi_x^n, \xi_x^n\right) \\
&\quad \geq \frac{1}{2}\left[(\xi^n, \xi^n) - (\xi^{n-1} \circ X_1^n \gamma^n, \xi^{n-1} \circ X_1^n \gamma^n)\right] + \Delta t\left(\varepsilon \xi_x^n, \xi_x^n\right) \\
&\quad \geq \frac{1}{2}\left[(\xi^n, \xi^n) - (\xi^{n-1}, \xi^{n-1})(1 + K\Delta t)\right] + \Delta t\left(\varepsilon \xi_x^n, \xi_x^n\right),
\end{aligned} \tag{25}$$

where we have used the same technique as in [24] and the definition of γ^n again.

By use of Lemma 4 and Lemma 5, we easily get

$$\begin{aligned}
A\left(\eta^n, \xi^n\right) &\leq \|\eta^n\|_A \|\xi^n\|_A \\
&\leq K\left(\frac{h^2}{H^2} + \frac{\Delta t}{H^2}\right)^{\frac{m}{2}} \sqrt{\Delta t}\left[1 + \|\xi^{n-1}\|^2_{H^1(\Omega)}\right]^{1/2} \|\xi^n\|_A \\
&\leq K\Delta t\left(\frac{h^2}{H^2} + \frac{\Delta t}{H^2}\right)^m \left[1 + \|\xi^{n-1}\|^2_{H^1(\Omega)}\right] + \delta\|\xi^n\|_A^2.
\end{aligned} \tag{26}$$

Substituting (25) and (26) into (24), we obtain

$$\begin{aligned}
&\frac{1}{2}\left[(1-2\delta)(\xi^n, \xi^n) - (\xi^{n-1}, \xi^{n-1})\right] + \frac{\varepsilon}{2}(1-2\delta)\Delta t\|\xi^n\|^2_{H^1(\Omega)} \\
&\quad \leq K\Delta t\left\{\|\xi^{n-1}\|^2_{L^2(\Omega)} + \left(\frac{h^2}{H^2} + \frac{\Delta t}{H^2}\right)^m \left(1 + \|\xi^{n-1}\|^2_{H^1(\Omega)}\right)\right\}.
\end{aligned} \tag{27}$$

Multiply (27) by 2 and sum it over time, then, for sufficiently small Δt and δ we have

$$\|\xi^n\|^2_{L^2(\Omega)} + \left(\sum_{i=1}^{n} \|\xi^i\|^2_{H^1(\Omega)} \Delta t\right)^{1/2} \leq K\left\{\left(\frac{h^2}{H^2} + \frac{\Delta t}{H^2}\right)^m + \Delta t \sum_{i=1}^{n-1} \|\xi^i\|^2_{L^2(\Omega)}\right\}. \tag{28}$$

Using the discrete Gronwall's lemma, we get

$$\max_n \|\xi^n\|^2_{L^2(\Omega)} \leq K\left(\frac{h^2}{H^2} + \frac{\Delta t}{H^2}\right)^m. \tag{29}$$

By Lemma 1, we get the inequality (8). So we complete the proof of Theorem 1. □

5 Numerical results

In this section, we present results of some numerical experiments with the main objective for confirming our theoretical analysis. Set $[a, b] = [0, 2]$, denote by H the overlapping degree, and then decompose the interval $[a, b]$ into the following three

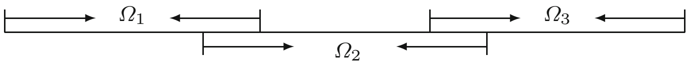

Fig. 1 The sub-domains of the interval $[a, b]$

Table 1 For the case: $\nu = \frac{1}{10}$, $H = \frac{1}{6}$, $h = \triangle t$

h	m	$\varepsilon = 1$	$\varepsilon = 1e-1$	$\varepsilon = 1e-3$	$\varepsilon = 1e-5$	$\varepsilon = 0$
		L^2	L^2	L^2	L^2	L^2
$\frac{1}{24}$	*	$2.2821e-2$	$2.8124e-2$	$3.4516e-2$	$3.4913e-2$	$3.4917e-2$
$\frac{1}{24}$	1	$4.2306e-2$	$3.0104e-2$	$3.1740e-2$	$3.2727e-2$	$3.2758e-2$
$\frac{1}{24}$	2	$3.7503e-2$	$2.8372e-2$	$3.1735e-2$	$3.2728e-2$	$3.2759e-2$
$\frac{1}{24}$	3	$3.5857e-2$	$2.8206e-2$	$3.1735e-2$	$3.2728e-2$	$3.2759e-2$
$\frac{1}{24}$	4	$3.4863e-2$	$2.8190e-2$	$3.1735e-2$	$3.2728e-2$	$3.2759e-2$
$\frac{1}{48}$	*	$1.2479e-2$	$1.4363e-2$	$1.5231e-2$	$1.5424e-2$	$1.5427e-2$
$\frac{1}{48}$	1	$1.9319e-2$	$1.4910e-2$	$1.4682e-2$	$1.4682e-2$	$1.4682e-2$
$\frac{1}{48}$	2	$1.7447e-2$	$1.4404e-2$	$1.4676e-2$	$1.4681e-2$	$1.4681e-2$
$\frac{1}{48}$	3	$1.6942e-2$	$1.4387e-2$	$1.4676e-2$	$1.4681e-2$	$1.4681e-2$
$\frac{1}{48}$	4	$1.6753e-2$	$1.4386e-2$	$1.4676e-2$	$1.4681e-2$	$1.4681e-2$
$\frac{1}{96}$	*	$6.6170e-3$	$7.2218e-3$	$7.3159e-3$	$7.3169e-3$	$7.3169e-3$
$\frac{1}{96}$	1	$8.0829e-3$	$7.3664e-3$	$7.3054e-3$	$7.3055e-3$	$7.3055e-3$
$\frac{1}{96}$	2	$7.3927e-3$	$7.2298e-3$	$7.3040e-3$	$7.3054e-3$	$7.3054e-3$
$\frac{1}{96}$	3	$7.2718e-3$	$7.2283e-3$	$7.3040e-3$	$7.3054e-3$	$7.3054e-3$
$\frac{1}{96}$	4	$7.2456e-3$	$7.2283e-3$	$7.3040e-3$	$7.3054e-3$	$7.3054e-3$

sub-domains: $\Omega_1 = [0, \frac{2}{3} + \frac{H}{2}]$, $\Omega_2 = [\frac{2}{3} - \frac{H}{2}, \frac{4}{3} + \frac{H}{2}]$ and $\Omega_3 = [\frac{4}{3} - \frac{H}{2}, 2]$ (see Fig. 1).

In this section, we choose the piecewise linear finite element spaces and apply the same linear partition functions of unity as in [20].

Experiment I In this experiment, we consider that the velocity field is a constant. The exact solution is $u = e^{-t} \sin^2 \pi x$ and the velocity is $\nu = 0.1$. Choosing different values of the parameters $\varepsilon, h, \triangle t$ and the iterative number m, we present the following errors of $\|u - u_h\|_{L^2}$ in Tables 1 and 2. In these Tables, we use " * " to denote the numerical results by using MCC-FE method. Based on these numerical results, it is easily seen that only one or two iterations are needed to reach the given accuracy for the fixed overlapping degree H at each time step, which is coincided with our theoretical analysis.

Experiment II Here we consider a variable velocity field. Take the exact solution $u = e^{-t} \sin \pi x$ and $\nu = \frac{x}{10}$. Choosing different values of the parameters $\varepsilon, h, \triangle t$ and the iterative number m, we present the following errors of $\|u - u_h\|_{L^2}$ in Table 3. These numerical results suggest that our theoretical analysis is valid.

Experiment III To compare the numerical results by MCC-FE method and Parallel algorithm, we will choose a right-hand side function f with complex structure and

Table 2 For the case: $\nu = \frac{1}{10}$, $H = \frac{1}{12}$, $h = \triangle t$

h	m	$\varepsilon = 1$	$\varepsilon = 1e-1$	$\varepsilon = 1e-3$	$\varepsilon = 1e-5$	$\varepsilon = 0$
		L^2	L^2	L^2	L^2	L^2
$\frac{1}{24}$	*	$2.2821e-2$	$2.8124e-2$	$3.4516e-2$	$3.4913e-2$	$3.4917e-2$
$\frac{1}{24}$	1	$3.5462e-2$	$3.0263e-2$	$3.0021e-2$	$3.3080e-2$	$3.3196e-2$
$\frac{1}{24}$	2	$2.9672e-2$	$2.7975e-2$	$3.0020e-2$	$3.3072e-2$	$3.3188e-2$
$\frac{1}{24}$	3	$2.5252e-2$	$2.7240e-2$	$3.0020e-2$	$3.3071e-2$	$3.3187e-2$
$\frac{1}{24}$	4	$2.2788e-2$	$2.7035e-2$	$3.0020e-2$	$3.3071e-2$	$3.3187e-2$
$\frac{1}{48}$	*	$1.2479e-2$	$1.4363e-2$	$1.5231e-2$	$1.5424e-2$	$1.5427e-2$
$\frac{1}{48}$	1	$1.6714e-2$	$1.5027e-2$	$1.4641e-2$	$1.4975e-2$	$1.5042e-2$
$\frac{1}{48}$	2	$1.4290e-2$	$1.4242e-2$	$1.4630e-2$	$1.4975e-2$	$1.5042e-2$
$\frac{1}{48}$	3	$1.2695e-2$	$1.4103e-2$	$1.4630e-2$	$1.4975e-2$	$1.5042e-2$
$\frac{1}{48}$	4	$1.1991e-2$	$1.4079e-2$	$1.4630e-2$	$1.4975e-2$	$1.5042e-2$
$\frac{1}{96}$	*	$6.6170e-3$	$7.2218e-3$	$7.3159e-3$	$7.3169e-3$	$7.3169e-3$
$\frac{1}{96}$	1	$7.9674e-3$	$7.4128e-3$	$7.2947e-3$	$7.2961e-3$	$7.2962e-3$
$\frac{1}{96}$	2	$7.3009e-3$	$7.1720e-3$	$7.2920e-3$	$7.2958e-3$	$7.2959e-3$
$\frac{1}{96}$	3	$6.8992e-3$	$7.1521e-3$	$7.2920e-3$	$7.2958e-3$	$7.2959e-3$
$\frac{1}{96}$	4	$6.9807e-3$	$7.1504e-3$	$7.2920e-3$	$7.2958e-3$	$7.2959e-3$

Table 3 For the case: $\nu = \frac{x}{10}$, $H = \frac{1}{6}$, $h = \triangle t$

h	m	$\varepsilon = 1$	$\varepsilon = 1e-1$	$\varepsilon = 1e-3$	$\varepsilon = 1e-5$	$\varepsilon = 0$
		L^2	L^2	L^2	L^2	L^2
$\frac{1}{12}$	*	$3.6687e-2$	$6.1793e-2$	$7.4209e-2$	$7.5258e-2$	$7.5270e-2$
$\frac{1}{12}$	1	$7.2542e-2$	$6.9213e-2$	$6.9410e-2$	$7.1682e-2$	$7.1718e-2$
$\frac{1}{12}$	2	$5.8469e-2$	$6.2958e-2$	$6.9400e-2$	$7.1676e-2$	$7.1713e-2$
$\frac{1}{12}$	3	$4.9055e-2$	$6.1716e-2$	$6.9399e-2$	$7.1676e-2$	$7.1712e-2$
$\frac{1}{12}$	4	$4.4928e-2$	$6.1469e-2$	$6.9399e-2$	$7.1676e-2$	$7.1712e-2$
$\frac{1}{24}$	*	$2.3748e-2$	$3.2185e-2$	$3.3497e-2$	$3.3510e-2$	$3.3510e-2$
$\frac{1}{24}$	1	$3.7178e-2$	$3.4425e-2$	$3.3328e-2$	$3.3324e-2$	$3.3324e-2$
$\frac{1}{24}$	2	$3.0658e-2$	$3.2371e-2$	$3.332e-2$	$3.3318e-2$	$3.3319e-2$
$\frac{1}{24}$	3	$2.7330e-2$	$3.2180e-2$	$3.3301e-2$	$3.3318e-2$	$3.3318e-2$
$\frac{1}{24}$	4	$2.6120e-2$	$3.2162e-2$	$3.3301e-2$	$3.3318e-2$	$3.3318e-2$
$\frac{1}{48}$	*	$1.3815e-2$	$1.6319e-2$	$1.6651e-2$	$1.6654e-2$	$1.6654e-2$
$\frac{1}{48}$	1	$1.8363e-2$	$1.6941e-2$	$1.6608e-2$	$1.6607e-2$	$1.6606e-2$
$\frac{1}{48}$	2	$1.5613e-2$	$1.6344e-2$	$1.662e-2$	$1.6606e-2$	$1.6606e-2$
$\frac{1}{48}$	3	$1.4665e-2$	$1.6323e-2$	$1.662e-2$	$1.6606e-2$	$1.6606e-2$
$\frac{1}{48}$	4	$1.4400e-2$	$1.6323e-2$	$1.662e-2$	$1.6606e-2$	$1.6606e-2$

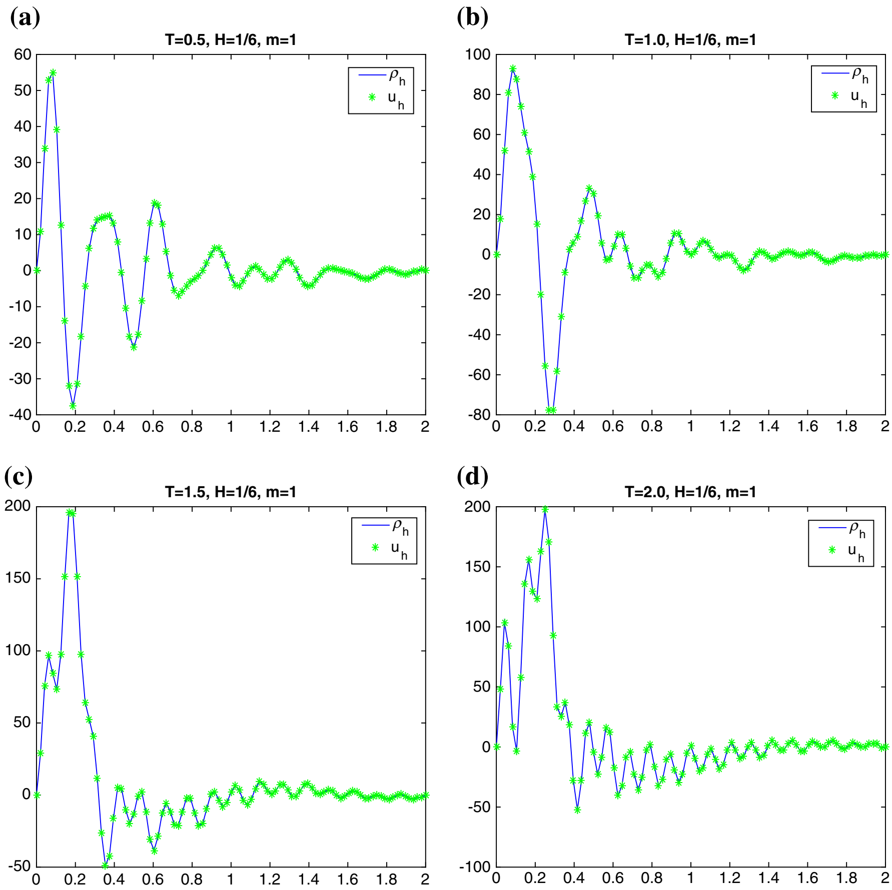

Fig. 2 Numerical results at different times under the case $\nu = 1/10$: $T = 0.5$ (**a**), $T = 1.0$ (**b**), $T = 1.5$ (**c**), $T = 2.0$ (**d**)

the homogeneous initial-boundary conditions as follows:

$$\begin{cases} f(x,t) = 100e^{t-\frac{x^2}{2}}\cos(5\pi xt)\sin(9\pi x), \\ u(a,t) = u(b,t) = 0, \\ u^0(x) = 0. \end{cases}$$

Here, we fix the parameters $H = 1/6$, $h = \Delta t = 1/48$, $\nu = 0.1$ and $\varepsilon = 1e-5$ and take iterative number $m = 1$, we observe numerical solutions at different time (see Fig. 2) where we use " - " to denote numerical results of ρ_h by MCC-FE method and use " * " to denote numerical results of u_h by Parallel algorithm. These figures clearly show that for the advection–dominated diffusion equation, u_h approximates well to ρ_h at different times, even only iterating one cycle at each time step under the constant velocity field.

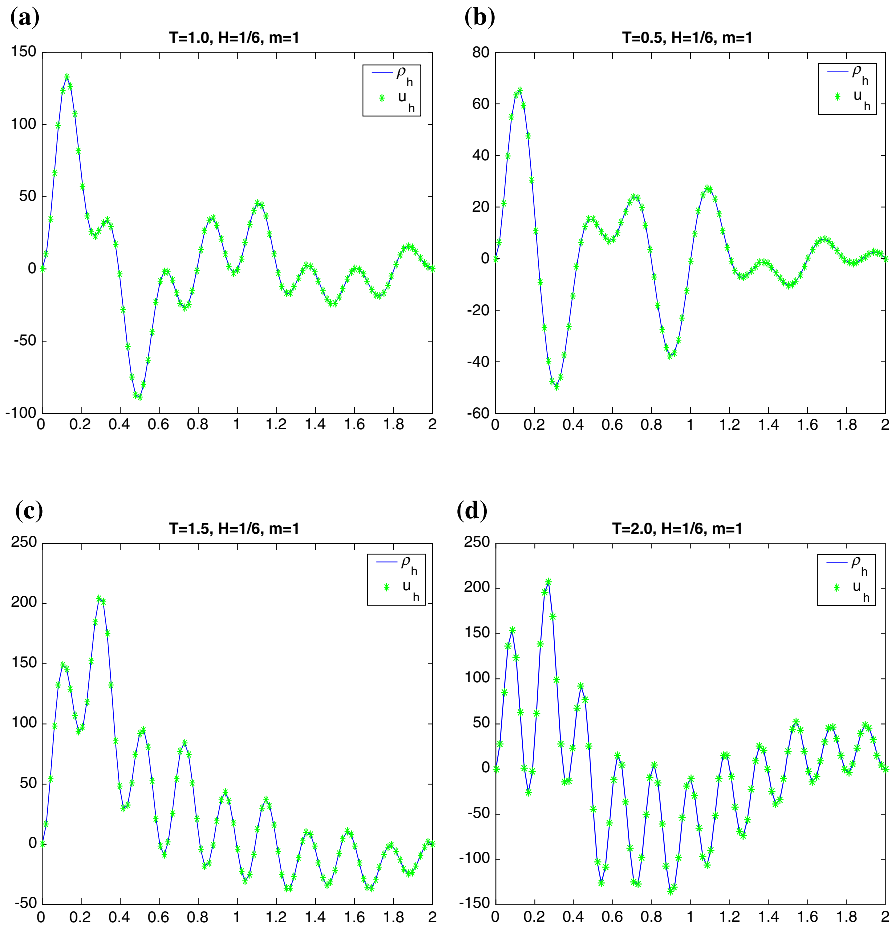

Fig. 3 Numerical results at different times under the case $\nu = e^{-x}/10$: $T = 0.5$ (**a**), $T = 1.0$ (**b**), $T = 1.5$ (**c**), $T = 2.0$ (**d**)

Experiment IV In this experiment, we compare the numerical results by MCC-FE method and Parallel algorithm under the variable velocity field. We still choose a right-hand side function f with complex structure and the homogeneous initial-boundary conditions as follows:

$$\begin{cases} f(x,t) = 100e^{x-\frac{x^2}{2}+t}\cos(3\pi xt)\sin(5\pi x), \\ u(a,t) = u(b,t) = 0, \\ u^0(x) = 0. \end{cases}$$

For the variable velocity $\nu = e^{-x}/10$, we fix the parameters $H = 1/6$, $h = \triangle t = 1/48$ and $\varepsilon = 1e-5$ and take iterative number $m = 1$, we observe numerical solutions at different times (see Fig. 3) . These figures clearly show that for the variable velocity field, u_h also approximates well to ρ_h at different times, even only iterating one cycle at each time step.

Acknowledgments The authors thank the reviewers for their helpful comments and suggestions, which greatly improved the quality of this article. Zhang's work was partially supported by the National Nature Science Foundation of China (11401588), the Natural Science Foundation of Shandong Province (ZR2014AQ005), China Scholarship Council and Young Talent Attraction fellowship from CNPq. Fu's work was partially supported by the National Nature Science Foundation of China (11201485).

References

1. Bramble, J.H., Pasciak, J.E., Xu, J.: Convergence estimates for product iterative methods with application to domain decomposition. Math. Comput. **57**, 1–21 (1991)
2. Bramble, J.H., Pasciak, J.E., Xu, J.: Parallel multilevel preconditioners. Math. Comput. **55**, 1–22 (1990)
3. Cai, X.C.: Some domain decomposition algorithms for nonselfadjont elliptic and parabolic partial differential equations. PhD. thesis, Courant Institute (1989)
4. Dryja, M., Widlund, O.B.: An additive variant of Schwarz alternating methods for many subregions. Tech. Report 339 Dept. of Comp. Sci. Coutant Institute. (1987)
5. Lu, T., Shih, T.M., Liem, C.B.: Two synchronous parallel algorithms for partial differential equations. J. Comput. Math. **9**, 74–85 (1991)
6. Xu, J.: The method of subspace corrections. J. Comput. Appl. Math. **128**, 335–362 (2001)
7. Xu, J.: Theory of multilevel methods. PhD thesis, Cornell University (1989)
8. Xu, J.: Iterative methods by space decomposition and subspace correction: a unifying approach. SIAM Rev. **34**, 581–613 (1992)
9. Cai, X.C.: Additive Schwarz algorithms for parabolic convection-diffusion equations. Numer. Math. **60**, 41–61 (1991)
10. Cai, X.C.: Multiplicative Schwarz methods for parabolic problem. SIAM J. Sci. Comput. **15**, 587–603 (1994)
11. Tai, X.C.: A space decomposition method for parabolic equations. Numer. Methods Part. Differ. Equ. **14**, 24–46 (1998)
12. Rui, H., Yang, D.P.: Schwarz type domain decomposition algorithms for parabolic equations and error estimates. Acta Math. Appl. Sin. **14**, 300–313 (1998)
13. Rui, H., Yang, D.P.: Multiplicative Schwarz algorithm with time stepping along characteristics for convection diffusion equations. J. Comput. Math. **19**, 501–510 (2001)
14. Wang, H., Liu, J.G., Espedal, M.S., Ewing, R.E.: A characteristic nonoverlapping domain decomposition method for multidimensional convection-diffusion equations. Numer. Methods Part. Differ. Equ. **21**, 89–103 (2005)
15. Yang, D.P.: Parallel domain decomposition procedures of improved D-D type for parabolic problems. J. Comput. Appl. Math. **233**, 2779–2794 (2010)
16. Ma, K.Y., Sun, T., Yang, D.P.: Parallel Galerkin domain decomposition procedure for parabolic equation on general domain. Numer. Methods Part. Differ. Equ. **25**, 622–636 (2009)
17. Dawson, C.N., Dupont, T.F.: Explicit/implicit conservative Galerkin domain decomposition procedures for parabolic problems. Math. Comput. **58**, 21–34 (1992)
18. Zhang, J.S., Yang, D.P.: Parallel least-squares finite element method for time-dependent convection-diffusion problem. Computing **91**(3), 217–240 (2011)
19. Zhang, J.S., Yang, D.P.: Parallel characteristic finite difference method for time-dependent convection-diffusion equations. Numer. Methods Part. Differ. Equ. **27**(4), 854–866 (2011)
20. Zhang, J.S., Yang, D.P., Fu, H.F., Guo, H.: Parallel characteristic finite element method for time-dependent convection-diffusion problem. Numer. Linear Algebra Appl. **18**(4), 695–705 (2011)
21. Wang, H., Rui, H., Zhang, S.: An optimal-order error estimate for the mass-conservative characteristic finite element scheme. Appl. Math. Comput. **218**, 10271–10278 (2012)
22. Rui, H., Tabata, M.: A mass-conservative characteristic finite element scheme for convection-diffusion problems. J. Sci. Comput. **43**, 416–432 (2009)
23. Rui, H., Tabata, M.: A second order characteristic finite element scheme for convection-diffusion problems. Numer. Math. **92**, 161–177 (2002)

24. Douglas Jr, J., Russell, T.F.: Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures. SIAM J. Numer. Anal. **19**, 871–885 (1982)